Partial vs. Full Mixed Bundling of Digital Goods

Hemant K. Bhargava*
Graduate School of Management
University of California Davis
hemantb@gmail.com

March 11, 2014

Abstract

The distribution and consumption of media, entertainment, software, and other information goods has transformed in recent years. Consumers demand access on multiple devices, and content providers have responded with different strategies, ranging from independent pricing on each device to a single price for access to all devices. I frame the multi-device design problem as a choice between pure bundling (one price gets both devices), mixed bundling (price each device separately, and offer discount for getting both) and partial bundling (one device is sold separately and also bundled with the second). When one device is considered superior by all customers, then multi-device discounts help the firm if higher-value customers have greater propensity for multi-device access; the choice of bundling strategy depends on certain ratios of valuations and contingent valuations for the two devices. When consumer valuations for the traditional and emerging devices are mutually independent, then a full mixed bundling is optimal when the demand profiles for the two devices are relatively similar; otherwise, it is optimal to sell bundle access to the weaker device into the price for the superior device while also selling the weaker device separately. When devices behave more like substitutes, then such partial bundling is less likely to be optimal.

*Many thanks to colleagues who generously read and provided useful feedback on earlier drafts, especially D.J. Wu, Olivier Rubel, Amanda Kimball and Manish Gangwar.

Partial vs. Full Mixed Bundling of Digital Goods

1 Introduction

The distribution and consumption of media, entertainment, software, and other forms of information goods has been transformed in the last two decades. Digital content such as *Netflix* streaming and *Kindle* ebooks, previously accessed on television sets and computers, is now consumed on smartphones and tablets. Software previously installed on enterprise data centers now runs over the Internet cloud (e.g., *Adobe* Creative Cloud). Print products are now distributed and consumed digitally. For instance, traditional readers of *The Economist* or *Wall Street Journal* value accessing these products online despite having the print version. The transformation to multi-device consumption accelerated with the emergence and widespread adoption of the Internet and World Wide Web in the last two decades. More recently, the "anytime, anywhere" computing enabled by the proliferation of smartphones and tablets has again disrupted established patterns for consuming information, entertainment, and computing products. Another transformation is around the corner in the form of wearable computers such as eyeglasses and wrist watches.

Multi-device consumption of digital goods presents content producers with a product design and management problem. Should they charge once for the content and allow access across any and all devices? Should they charge separately and multiple times? Should they offer bundle discounts for multi-device access? Should they offer a single-device price for every device or only some? This problem is naturally analyzed using the economic lens of product bundling, which has been studied extensively since the early work by Stigler (1963) and Adams and Yellen (1976). The mapping to bundling theory is straightforward. Two distinct component products 1 and 2 (here, *access* to content via devices 1 and 2) can either be sold and priced individually (*unbundled separate sales* at prices p_1, p_2 respectively), only as a bundle at price p_B (*pure bundling*), or both individually and as a bundle (*mixed bundling*);

the last also admits the special case of *partial bundling* where only one component is sold separately, in addition to the bundle. As depicted in Fig. 1, the relation between the three prices reflects the selling strategy in use.

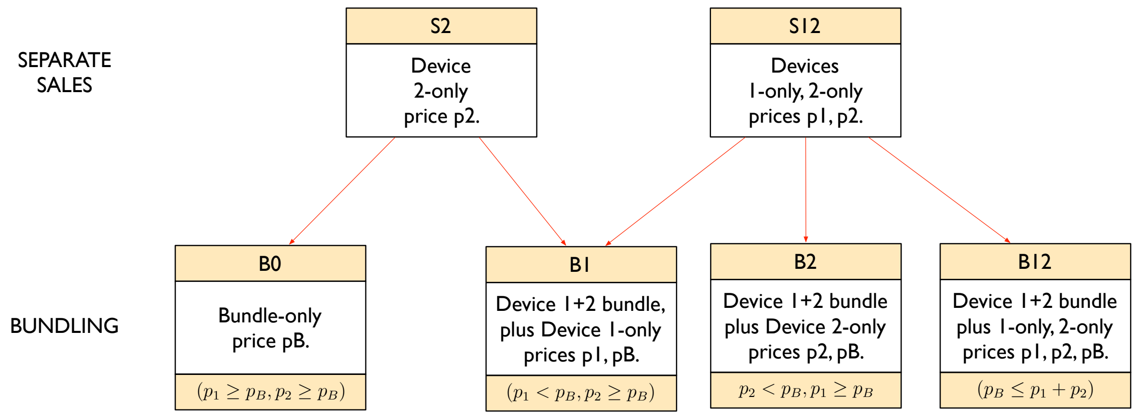


Figure 1: Strategies for providing access on multiple devices. device 1 is the one with a weaker demand profile—usually this is the emerging device—while device 2 with the stronger profile tends to be the traditional device.

However, the multi-device problem has some fundamental characteristics that push the envelope of analytical and qualitative results in bundling theory. First, because the same content is accessed on multiple devices, users' willingness to pay for multi-device access is typically lower than the sum of their willingness to pay for each single device. Second, there might be non-zero dependence between consumers' valuations across multiple devices, with both positive and negative dependence across different applications. Third, when one device has a primary role for consumption relative to other devices, the devices might not be symmetric in their single-device valuations. In contrast, the majority of bundling literature employs additive valuations, independent distributions, and symmetric single-device demand. This paper generalizes the study of bundling on these three dimensions, and in doing so develops a series of practical insights regarding multi-device bundling. §**??** elaborates on the linkages between this paper and the model structures and results in existing bundling literature.

The questions asked in this paper are both highly relevant and timely with regard to pricing of digital goods. Many content providers offer access via multiple devices, but they differ widely in how they have deployed multi-device access.[1] For instance, *The New York Times* offers a *pure bundle* where free and full access to the online edition is rolled into its print subscription. *The Economist* employs *mixed bundling*: users can subscribe to either just print or just digital, or to a discounted bundle of print and digital access. The *Wall Street Journal* employs a *partial* mixed bundling—or *tying*—strategy by pricing a digital-only edition and a digital+print bundle, but with no separate price for a digital-only product; thus, the print version is tied to the digital version (the tying good). Indeed, *The Economist* employed this approach until December 2012, before adopting mixed bundling (Ives, 2012). Software vendors can either charge separate prices for each access device, or offer multi-device access for a bundle price, especially under a device-agnostic cloud-based computing model. Content aggregation and distribution firms such as *Comcast* have traditionally limited subscribers to the traditional consumption device (television), but Comcast's Airplay now extends the TV subscription to Internet devices for an additional price. In contrast, *Netflix* allows subscribers to access their account from any supporting device at no additional cost (a pure bundle). The premium content provider *HBO* also gives its TV subscribers free additional access on tablets and other devices (under the *HBO GO* label) at no additional cost; but *HBO GO* is not sold separately (implying that the firm sacrifices selling opportunities to the tens of millions of consumers who do not buy the cable or satellite package), though this possibility is being actively considered by the company (Wallenstein, 2013).

This proliferation of selling strategies motivates the fundamental question analyzed in this paper: *how should content providers manage, and price, multi-device demand?* I develop a model to analyze alternative strategies under a spectrum of market characteristics and consumer preferences for the multiple access devices and devices, and I study the nature of information required to implement the more "advanced" strategies involving the use of bundling. I describe how the choice of selling strategy is influenced by alternative relation-

[1] I note that while the product being sold is *access to content via a certain device*, it is convenient to write as if it is the *device* that is being sold.

Strength of Platform Demand Profiles	Positive correlation, Vertical differentiation	Zero correlation, Independent Demand	Negative correlation, Horizontal differentiation
Equal or Similar	NA / No benefit from multi-platform discount.	Full mixed bundling.	Full mixed bundling.
Highly Dissimilar	Partial bundling {1,B} or {2,B} under low bundle disutility. O/w Full mixed or no bundling.	Partial bundling {1,B} if bundle disutility is low. O/w full mixed bundling.	Partial bundling {1,B} Independent pricing if bundle disutility is high.

Table 1: Summary of insights regarding choice of multi-device strategy.

ships between a consumer's valuation of the product under the traditional device vs. their valuation under the new or emerging device. The choice of strategy is a salient question because, in absence of other guidelines, many content providers have simply adopted a *tying* strategy in which subscription under the traditional device is a prerequisite to gain access under the new device. Needless to say, the problem is more nuanced, and this paper seeks to provide practical guidance regarding optimal multi-device strategy under different conditions. Table 1 provides a simple summary of the resulting guidelines, while the detailed and nuanced results are presented in §4 which analyzes the multi-device strategy by successively examining the three cases of positively correlated single-device demand, independent demand, and negatively correlated demand. §3 describes the modeling framework and discusses related studies of multi-device strategy. §5 summarizes results across the multiple cases and discusses contributions, limitations, and future directions.

2 Related Literature

The earliest papers on bundling established the key economic reason behind bundling, that it reduces the across-consumer dispersion in product valuations (Stigler, 1963; Adams and Yellen, 1976; Schmalensee, 1984; McAfee et al., 1989). Later papers have highlighted additional advantages, including supply-side economies of scope (Evans and Salinger, 2005; Suroweicki, 2010), lower consumer transaction costs or other demand-side conveniences and network effects (Lewbel, 1985; Prasad et al., 2010), and strategic leverage across products (Burstein, 1960; Carbajo et al., 1990; Eisenmann et al., 2011; Stremersch and Tellis, 2002). Because the literature on bundling is so vast, this section limits attention to elements of bundling theory that are most relevant to the setting studied in this paper. For a broader perspective, readers can turn to Stremersch and Tellis (2002), Kobayashi (2005) and Venkatesh and Mahajan (2009) which provide excellent overviews of the literature on bundling.

Bundling Component Goods with Correlated Demand

Schmalensee (1984) modeled two component goods with bivariate Gaussian demand. The model allowed for non-zero correlation among component demands, establishing that i) negative correlation among the two demands was *not* a necessary condition for bundling to increase profit, and ii) mixed bundling "serves as a powerful price discrimination device". Because Gaussian demand does not yield closed-form optimal solutions, the main results were demonstrated numerically and the comparison was limited to pure bundling vs. unbundled sales and mixed bundling vs. unbundled sales. Moreover, this analysis was restricted to additive valuations ($v_B = v_1 + v_2$) and symmetric distributions. While maintaining these restrictions, Long (1984) and McAfee et al. (1989) extended the analysis to arbitrary demand distributions and demonstrated that mixed bundling increases profits nearly always, including under oligopolistic competition. Salinger (1995) confirmed the role of correlation using graphical analysis, and highlighted the incentives to bundle under positive correlation when bundling has a supply-side economic benefit. A more rigorous treatment of correlation is found in Chen and Riordan (2013)'s recent work applying copula functions (which capture density of the joint distribution, hence allowing arbitrary levels of correlation) to the analysis of bundling two component goods. The analysis generalizes prior results regarding the effect of correlation on the attractiveness of bundling. None of these papers considers non-additive valuations or the possibility that partial mixed bundling might be superior to full mixed bundling when the component goods are not symmetric. Hence these prior results cannot be applied to shed insights regarding the optimal bundling strategies for multi-device consumption.

Bundling of Complementary or Substitutable Goods

Lewbel (1985) recognized that consumers might perceive the components of a bundle as substitutes and complements, hence have sub-additive or super-additive valuations for the bundle. The paper proves by example that bundling (conversely, unbundled sales) can be the optimal strategy even when component goods are substitutable (conversely, complementary), and that mixed bundling improves profits via a price discrimination mechanism. In unpublished work on non-additive valuations, Ibragimov (2005) finds that bundling is preferred in the case of complementary goods and heavy tailed distribution, while unbundling is best when valuations are not extremely heavy-tailed, and when the products are substitutes. The model allows for large number of component goods, with independent and symmetric marginal demand functions. While these papers produce some insights on bundling of substitutable goods, these results cannot be readily applied to the present problem because of the model restrictions (independence and symmetry) and absence of partial bundling strategies.

Bundling Goods with Asymmetric Demand Functions

As noted above, the majority of bundling literature considers bundling of two (or more) goods that have similar demand profiles (i.e., a random consumer's valuation for any component is a draw from the same distribution). In some papers, the effect of asymmetric demand is discussed, but usually as an afterthought (Long, 1984; Bakos and Brynjolfsson, 1999) and without a consideration of how asymmetry impacts the choice of optimal bundling strategy (pure vs. partial vs. full mixed bundling). Fang and Norman (2006) consider non-symmetric component goods, and demonstrate the advantage of bundling large but finite number of goods (i.e., they demonstrate the variance-reduction effect of bundling without resorting to asymptotic proerties). However, the restriction to additive valuations and pure bundling (mixed or partial bundling is not considered) provides little insight with respect to the optimal bundling strategy for multi-device bundles. Bhargava (2013) identifies conditions for full vs. partial mixed bundling when a firm has two component goods with asymmetric demand functions. However, this analysis has limited application for the present problem because it is restricted to additive valuations and independent demand distributions.

Additional Research on Bundling

A few papers on bundling have investigated the role of more than one of the three dimensions identified above. Venkatesh and Kamakura (2003) develop a two-component model that allows both non-zero correlation and non-additive valuations, and analyzes both pure and mixed bundling. However, the analysis is primarily numerical (and an incorrect key result leaves some doubt about the analysis), does not characterize conditions for the optimal bundling strategy, and partial bundling is not considered because the component goods have symmetric demand profiles. Bakos and Brynjolfsson (1999) demonstrated the advantages of bundling large numbers of component goods especially when these goods have zero marginal costs. While the basic model has independent demands and additive valuations, they also comment on the role of correlation, complementarity and substitutability. However, the large-N focus forces the analysis to treat all these effects uniformly across all goods (e.g., if two goods are complementary, then all bundles of any size m also have a complementarity property and with the same complementarity factor) and the marginal or single-good demands to be identical.[2] Finally, Armstrong (2013) develops a general model of two-component bundles which allows for correlation and non-additive valuations, but partial bundling is not considered because the component goods are assumed to have symmetric demand. While this paper characterizes the optimality of bundling in terms of fundamental

[2] The fundamental construct for modeling consumer demand in Bakos and Brynjolfsson (1999) is that a consumer ω's valuation for good i, if she buys n goods, is $v_{ni}(\omega)$. Thus the valuation is dependent only on the number—rather than the specific subset—of goods purchased. A complementarity or substitutability factor α is introduced (α is independent of specific goods) to allow for non-additive valuations: $v_{ni}(\omega)=n^{\alpha}v_{11}$. As is evident from setting n to 1, this implies that a consumer has identical valuation v_{11} for every good when that good is purchased alone (i.e., marginal demands are identical).

demand constructs such as relative elasticity for component vs. bundle demands, the highly general results cannot be employed to deliver specific insights regarding the design of the optimal bundling strategy.

3 Model

The examples given in §1 span the space of bundling strategies—pure bundling, mixed bundling, partial bundling, and unbundled sales—validating the application of the bundling framework to the multi-device access design question. To introduce the framework formally, let p_1, p_2 be the prices for purchasing device i-only access (if offered), and let p_B be the price for bundled access (if offered, and with $p_B \leq p_1 + p_2$) . The firm employs a multi-device bundling strategy when it offers a positive bundle discount $\delta = p_1 + p_2 - p_B > 0$. Otherwise, when $p_B \geq p_1 + p_2$ then the firm makes no effort to encourage bundle consumption, and the firm's strategy is labeled independent pricing. It will often be useful to describe the firm's strategy in terms of p_1, p_2 and δ (rather than p_B).

3.1 Consumer Valuations

For consumer x, let $v_i(x)$ $(i = 1, 2)$ denote her standalone valuation for accessing the content via device i-only. If the consumer does not already possess the device and must incur a fixed cost for acquiring it(e.g., buying a smartphone or a gaming console), then v_i is considered to be net of the amortized or per-period fixed cost. For parsimony in notation the (x) qualifier will be dropped, unless required for clarity. Let $g(v_1, v_2)$ represent the joint distribution of single-device valuations, which is common knowledge to consumers and the firm. Let f_i denote the marginal or single-device densities, continuous on the interval $[0, a_i)$ (with cumulative distribution F_i, so that standalone demand for device i at price p is $D_i(p) = 1 - F_i(p)$). Both single-device demand functions have non-decreasing elasticity, i.e., the ratio $\epsilon(p) = -\frac{pD_i'(p)}{D_i(p)} = \frac{\mathrm{pf}_i(p)}{1 - F_i(p)}$ is non-decreasing in p. Without loss of generality, arrange consumers such that $v_2(x)$ is weakly increasing in x (i.e., $v_2'(x) \geq 0$), and where the label 2 is assigned to the device with superior demand profile, i.e., $\int_x v_2(x)\mathtt{d}x \geq \int_x v_1(x)\mathtt{d}x$ (with $a_2 \geq a_1$).[3] Generally, therefore, device 2 would correspond to the *traditional* device (usually, print or television or computer) because it tends to have the higher range of consumer valuations relative to the emerging device (e.g., smartphone). Sometimes, the reverse might be true; for instance, *Cosmopolitan* magazine recently set a higher price for its Tablet edition than for a print subscription (Hagey, 2013). The model is equally applicable to such reverse cases, and the reader can interpret device 2 simply as being the one with a higher range of valuations.

Let $v_B(x)$ be the valuation for multi-device access. By definition and allowing for free disposal of unwanted components, $v_B(x) \geq \max\{v_1(x), v_2(x)\}$. Then at any triplet of prices p_1, p_2, p_B (such that $p_i \leq \min\{a_i, p_B\}$ and $p_B \leq p_1 + p_2$) the market splits into 4 segments representing consumers who purchase no access, access under only one device (1 or 2), or both devices. Let $\mathcal{N}_i$ $(i = 1, 2)$ be the number of consumers who purchase device i-only, and $\mathcal{N}_B$ be the ones who buy both. A majority of the bundling literature studies the case where single-component valuations are additive, i.e., $v_B = v_1 + v_2$. Then, the sales levels $\mathcal{N}_i$ and $\mathcal{N}_B$ are computed as follows (where j is $3-i$).

$$\mathcal{N}_i = \int_0^{p_B - p_i} \int_{p_i}^{a_i} g(v_1, v_2)\mathtt{d}v_i \mathtt{d}v_j \tag{1a}$$

$$\mathcal{N}_B = \int_{p_B - p_1}^{a_2} \int_{p_B - p_2}^{a_1} g(v_1, v_2)\mathtt{d}v_1 \mathtt{d}v_2 - \int_{p_B - p_2}^{p_1} \left(\int_{p_B - p_1}^{p_B - x} g(v_1, v_2)\mathtt{d}v_2 \right) \mathtt{d}v_1. \tag{1b}$$

3.2 Disutility and Propensity for Multi-device Access

In general, however, consumers may perceive multiple devices as partial *substitutes*, because they offer access to the same content, albeit with different user interface features or other

[3] Despite this ordering on the aggregate demand profile, a *specific* consumer may still have a higher value for access via device 1. More precisely, the superior device could be defined as the one that would produce higher profit if the firm offered access only under one device.

details. Then, v_B will be below $v_1 + v_2$.[4] Let $\tilde{v}_i \leq v_i$ be a consumer's *contingent* valuation for device i, which is the valuation contingent on having device j. The gap $\eta(x) = \eta = (v_1 + v_2 - v_B) = (v_i - \tilde{v}_i) = (v_j - \tilde{v}_j)$ is a *disutility* of dual consumption faced by the consumer (shortened in the text as "bundle disutility"). Then, $\tilde{v}_1 = v_B - v_2$ and $\tilde{v}_2 = v_B - v_1$. Because of the bundle disutility, Eq. 1 no longer works, and the sales level of each product is computed as follows.

$$\mathcal{N}_i = \text{device } i\text{-only} \equiv \{x\colon\ (v_i - p_i) \geq \max\{0, (v_j - p_j), v_1 + v_2 - (p_B + \eta)\}\} \tag{2a}$$

$$\mathcal{N}_B = \text{multi-device} \equiv \{x\colon\ (v_1 + v_2 - (p_B + \eta)) \geq \max\{v_1 - p_1, v_2 - p_2, 0\}\}. \tag{2b}$$

The magnitude of disutility depends on product features, usage characteristics, or consumer preferences. For instance, bundle disutility for multi-device access to music may be low because users separately gain value from access while traveling, taking a walk, or sitting in an office. An ebook novel may present a high disutility because it is generally read only once. While the literature on bundling is vast, only a few papers provide an in-depth analysis under a positive (or negative) disutility. Lewbel (1985) is, to my knowledge, the earliest, but primarily provides comparative statics results around price deviations. Matutes and Regibeau (1992) examine compatibility between bundle components that provide complementarities. Venkatesh and Kamakura (2003) study substitutes and complements using numerical simulations. Armstrong (2013) employs an elegant theoretical analysis of substitutes and complements while identifying conditions for optimality of bundle discounts.

For the general case where $v_B \neq v_1 + v_2$, consumers' attitudes towards multi-device access can be summarized via two related concepts: the disutility of dual consumption and, conversely, the increase in valuation from moving from single-device to multi-device access.

[4] Venkatesh and Chatterjee (2006) report an empirical study in support of this observation. However, if the two devices have synergistic features, then consumers might view them as complements, with $v_B > v_1 + v_2$. Analysis of this case would be no different than the case $v_B < v_1 + v_2$ (the important consideration is to go beyond the "simple" case of additive valuations). I restrict the discussion and analysis in this paper to the substitutes case in order to maximize clarity and focus and to avoid unnecessary tedium.

The latter term is defined as a proportion of the single-device valuation.

Definition 1 (Consumer attitude towards multi-device access)**.** For consumer x,

- Disutility of dual consumption of devices is $\eta(x) = v_1(x) + v_2(x) - v_B(x)$.
- Propensity for multi-device access is $\psi(x) = \frac{v_B(x) - v_i(x)}{v_i(x)} = \frac{\tilde{v}_j(x)}{v_i(x)}$.

3.3 Marginal and Joint Distribution of Valuations

The formulation and theoretical results of bundling models depend critically on the distribution of dependencies between a consumer's valuations for the component goods. From a practical perspective, variations on this dimension map the model onto different application scenarios for multi-device access. I will examine carefully chosen points across the space of possible formulations, corresponding to zero, positive and negative correlation. While these designs are not exhaustive, systematic comparison and contrast across these designs will yield general insights regarding how the inter-relationship between demand preferences impacts the multi-device access strategy offered by the producer.

Zero correlation is the neutral case where valuation under the traditional device provides, probabilistically, no information regarding valuation under the emerging device. Negative correlation reflects products for which the traditional user base is quite conservative with respect to changing technology or delivery mechanisms. It might apply when the traditional device uniquely retains features that are essential for higher-end consumption. For instance, consider a software product with multiple complex menus and rich multimedia displays in multiple windows. A new smartphone interface for this product may attract previous non-users but would have little value to high-value traditional users. Positive correlation, in contrast, is likely when the traditional user base embraces the new or emerging device or views it as an extension, e.g., for news reports and magazines, or for software products with relatively simple input-output displays. It might also describe specialized devices carried by personnel such as sales staff, delivery staff, or health care professionals. A smartphone or tablet interface provides additional opportunities to consume the content.

3.4 Framework for Analysis of Bundle Discount

A multi-device strategy is one that employs a positive bundle discount to encourage multi-device sales. The device pricing problem can thus be posed as an optimization problem in the three prices p_1, p_2, p_B, with the sales levels $\mathcal{N}_i$ and $\mathcal{N}_B$ as defined in Eq. 2a–2b. The firm has a constant marginal cost c_i for access via device i,with $c_B = c_1 + c_2$. Because marginal costs have a rather predictable effect on outcomes, I will often drop them to avoid making the expressions tedious.

$$\begin{aligned} \underset{p_1,p_2,p_B}{\text{Maximize}} \quad \Pi &= (p_1 - c1)\mathcal{N}_1 + (p_2 - c_2)\mathcal{N}_2 + (p_B - c_B)\mathcal{N}_B \\ &= (p_1 - c1)(\mathcal{N}_1 + \mathcal{N}_B) + (p_2 - c_2)(\mathcal{N}_2 + \mathcal{N}_B) - \delta\mathcal{N}_B \\ \text{s.t.} \quad c_1 \leq p_1 \leq p_B, &\quad c_2 \leq p_2 \leq p_B, \quad c_B \leq p_B \leq p_1 + p_2. \end{aligned} \tag{3}$$

Many insights regarding multi-device strategy design can be framed in terms of boundary conditions for the optimization problem Eq 3. Specifically, when bundle disutility $\eta=0$ for all consumers, mixed bundling is relevant exactly when the problem has an interior solution in which $p_B<p_1+p_2$ (besides $p_1, p_2<p_B$). Similarly, pure bundling is practiced when the optimal solution is at the boundary $p_1=p_2=p_B$, while the two partial bundling strategies occur when the optimal solution is at the respective boundary $p_i=p_B$ (with $p_j<p_B$). Applications where bundling strategy is irrelevant (and separate selling is sufficient) map to conditions under which the optimal solution to the problem occurs at the boundary $p_B = p_1 + p_2$.

The firm can influence $\mathcal{N}_i$ and $\mathcal{N}_B$ by altering either single-device prices or the bundle discount. A small drop in p_i will a) convert some non-buyers into single-device buyers (for device i), b) convert some device-j buyers into device-i buyers (these are customers just at the margin of $(v_i - p_i) \leq (v_j - p_j)$), and c) convert some device-j buyers into multi-device buyers (these are customers at the margin of $(v_1 + v_2 - \eta - p_1 - p_2) \leq (v_j - p_j)$). Thus a drop in p_i will increase $\mathcal{N}_i$ and $\mathcal{N}_B$ and decrease $\mathcal{N}_j$ as well as $\mathcal{N}_j + \mathcal{N}_B$. Second, a small discount $\delta > 0$ would increase aggregate sales of device i by converting device j-only buyers at the margin given in (c) above into multi-device customers. Hence a small bundle discount affects aggregate sales of device i (i.e., $\mathcal{N}_i + \mathcal{N}_B$) just the same as a small drop in p_j increases multi-device sales. Formally,

$$\frac{\partial(\mathcal{N}_i + \mathcal{N}_B)}{\partial\delta} = -\frac{\partial\mathcal{N}_B}{\partial p_i} \quad \text{at } \delta = 0. \tag{4}$$

The profitability impact of a bundle discount can be computed using the elegant apparatus developed by Long (1984) and Armstrong (2013). Differentiating the firm's profit function (Eq. 3) at $\delta = 0$ (independent pricing) and applying Eq. 4 to this derivative, yields that a multi-device strategy should be adopted when

$$\left.\frac{\partial\Pi}{\partial\delta}\right|_{\delta=0} = \left.\left(-(p_1 - c_1)\frac{\partial\mathcal{N}_B}{\partial p_1} - (p_2 - c_2)\frac{\partial\mathcal{N}_B}{\partial p_2} - \mathcal{N}_B\right)\right|_{\delta=0} > 0. \tag{5}$$

Further rearrangement using a common variable t to act as a proxy for price adjustments to both p_1 and p_2 reveals an intuitive condition for the profitability of a bundle discount (please see Appendix for the formal proof of all results).

Lemma 1. Offering a small bundle discount at the best single-device prices (p_1, p_2) is profitable when elasticity of multi-device demand at those prices exceeds 1. Formally,

$$\left.\frac{\partial\Pi}{\partial\delta}\right|_{\delta=0} > 0 \equiv \left.\left(-\frac{t}{\mathcal{N}_B}\frac{\mathrm{d}\mathcal{N}_B(c_1 + t(p_1 - c_1), c_2 + t(p_2 - c_2))}{\mathrm{d}t}\right)\right|_{t=1} > 1. \tag{6}$$

3.5 Benchmark: Single-device Strategy

Suppose that all consumers have positive value for at most one device (i.e., $v_1(x)v_2(x) = 0$ for all x) hence each consumer's multi-device access valuation v_B is simply the maximum of her v_1 and v_2 valuations. Trivially, there is no benefit from bundling (either pure or mixed) because profit from any bundle price p_B would be exceeded by setting p_i's at the same level. The firm therefore practices a single-device strategy. With price p_i for device i, there is a unique indifference point (marginal consumer) $\hat{x}_i$ such that a fraction $(1 - F(\hat{x}_i))$ of the market purchases access. Using standard price optimization techniques, the optimal value of $\hat{x}_i$ is where the demand elasticity equals 1 (assuming zero marginal cost). Hence, for the two cases of selling either device 1-only or device 2-only, these indifference points are

$$\hat{x}_1 = \text{Sol.}\left[\frac{\text{f}(x)}{1-F(x)} \times \frac{v_1(x)}{\frac{\partial}{\partial x}\left(v_1(x)\right)} = 1\right] \tag{7}$$

$$\hat{x}_2 = \text{Sol.}\left[\frac{\text{f}(x)}{1-F(x)} \times \frac{v_2(x)}{\frac{\partial}{\partial x}\left(v_2(x)\right)} = 1\right]. \tag{8}$$

Because $p_i = v_i(x)$ for the marginal consumer, the firm's optimal profit in the two cases are, respectively, $(1-F(\hat{x}_1))v_1(\hat{x}_1)$ and $(1-F(\hat{x}_2))v_2(\hat{x}_2)$. The above terms can easily be extended to the case of constant positive marginal costs, simply by replacing the valuation v_i with the surplus from trade $(v_i - c_i)$. The $\hat{x}_i$ terms will frequently be featured in later results about choice of multi-device strategy.

A second useful benchmark involves a further restriction that the devices are vertically differentiated, and every consumer has higher valuation for device 2 than for device 1. While the firm offers both devices, each consumer wants at most one, the one that offers maximum surplus net of price. There are now two indifference points separating device-2 buyers (highest x) from device-1 buyers and those lowest-x customers who purchase nothing. From the vertical differentiation literature (see, e.g., Bhargava and Choudhary, 2008, Eq. 5 and 12), the first indifference point $\hat{x}_{12}$ (which separates device-2 buyers from device-1 buyers)[5] is defined as

$$\hat{x}_{12} = \text{Sol.}\left[\frac{\text{f}(x)}{1-F(x)} \times \frac{v_2(x)-v_1(x)}{\frac{\partial}{\partial x}\left(v_2(x)-v_1(x)\right)} = 1\right], \tag{9}$$

while the second (which separates device-1 buyers from non-buyers) is exactly the $\hat{x}_1$ in Eq. 7. Positive sales occur for both devices when $\hat{x}_{12} > \hat{x}_1$, for which Bhargava and Choudhary (2008)

[5] There are several indifference points relevant to this analysis: $\hat{x}_1, \hat{x}_2, \hat{x}_{12}, \hat{x}_{1B}, \hat{x}_{2B}, \hat{x}_B$. My naming convention is that a single subscript represents the marginal consumer when the product menu has only one item (1, 2, or B; the choice is buying that product or nothing), while a dual subscript represents the consumer who is indifferent between the two menu options named in the subscript.

provide the condition

$$\frac{\partial}{\partial x}\left(\frac{v_1(x)}{v_2(x)}\right)\Big|_{\hat{x}_1} < 0 \quad \equiv \quad \frac{\partial}{\partial x}\left(\frac{v_2(x)}{v_1(x)}\right)\Big|_{\hat{x}_1} > 0 \tag{10}$$

which translates, approximately (and by extending the requirement over all x rather than just at $\hat{x}_1$), to: *high-v consumers have immensely higher value for the superior device.* When Eq. 10 fails, then the firm sells device 2 only (with indifference point $\hat{x}_2$ in Eq. 8), because adding device 1 to the mix fails to improve profit. While this condition is derived under a restriction of single-device purchase, it will also be shown to be operative for full mixed bundling of vertically differentiated goods (because this setting also asks some consumers to choose between device 1-only and 2-only).

4 Multi-device Strategy

As discussed in §1, many providers of digital information and entertainment goods face the multi-device bundling problem. Kannan et al. (2009) helped the National Academies Press with pricing the print and PDF forms of its publications; they used choice experiments to understand customer preferences and numerical optimization to compute optimal prices. But bundling models are notoriously hard to solve *analytically*, even without the complications introduced by correlated valuations, disutility of dual consumption, and devices with unequal demand profiles. Theoretical models of bundling often relax one or more of these conditions in order to preserve analytical tractability. Venkatesh and Chatterjee (2006) address multi-device product strategy via numerical analysis, despite requiring that single-device valuations be uniformly distributed and uncorrelated, and restricting every consumer to purchase no more than one device. Yet, as stated earlier, all of these factors—sub-additivity, correlation, and asymmetric demands—are relevant to studying the multi-device access problem in this paper. Koukova et al. (2008) empirically demonstrate that mixed bundling is viable and profitable for multi-device goods even in the presence of substitutability. Two notable aspects of the current paper are that it addresses the mixed bundling problem in full (i.e., covering full mixed bundling, partial bundling, pure bundling, and unbundled sales) and it tackles the analytical complexity with a divide-and-conquer approach that produces a series of insights by considering various fronts of the general problem. First up is designing a multi-device strategy when the devices are vertically differentiated.

4.1 Positive Correlation: Vertically differentiated devices

Suppose that all consumers consider one device (device 2) more attractive. Examples include iPad vs iPhone versions of an app, and software products which have rich multimedia displays and require complex manipulations. For instance, consider software applications from Adobe and Autodesk, both of which have strategically chosen to offer software as a service over the Internet. Most users of Autodesk's AutoCad or Adobe's Illustrator would obtain greater value from accessing the application via full-fledged computers rather than over smartphone and tablet devices. Since multi-device access provides higher valuation than any single device, we have $v_B > v_2 > v_1$ for all consumers. Without loss of generality, assume x's are ordered such that $v_1'(x) \geq 0$. Then the additional properties of $v_2'(x) \geq 0$ and $v_B'(x) \geq 0$ follow from the single-crossing property assumption which is common in the literature on vertical differentiation.

Assumption 1 (Single-Crossing Property)**.** With x's arranged such that $v_1'(x) \geq 0$, the incremental benefit from higher-valued devices is increasing in x. Formally, $\frac{\partial}{\partial x}(v_2(x) - v_1(x)) \geq 0$ and $\frac{\partial}{\partial x}(v_B(x) - v_i(x)) = \frac{\partial \tilde{v}_j}{\partial x} \geq 0$.

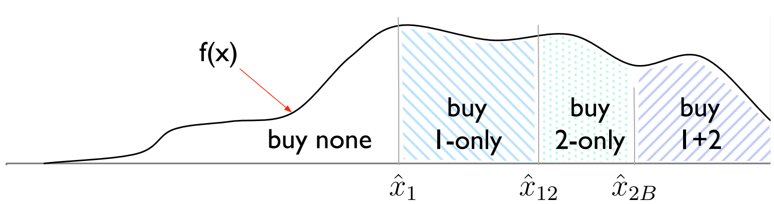


Figure 2: Indifference points and sales for vertically differentiated devices.

Let X_i be the set of consumers who purchases access under device i (either i alone or multi-device), given prices p_1, p_2, p_B. The single-crossing property ensures that at any triplet of prices, the highest-x consumers (but possibly none) will purchase the bundle; the next-highest (again, possibly none) purchase device 2 and then 1 in sequence; while the lowest-x consumers will purchase nothing (see Fig. 2). Of the different market share combinations implied by this, two can trivially be ruled out, that only device 1 is sold and that only device 2 is sold (trivially, selling the bundle would be more attractive). Hence the firm needs to choose whether, in addition to the bundle, it should offer both devices separately (mixed bundling), just one (partial bundling), or none (pure bundle). Standard incentive compatibility constraints (and the individual rationality constraint for device 1) yield the indifference points as solutions to the following equations,

$$v_B(\hat{x}_{2B}) - p_B = v_2(\hat{x}_{2B}) - p_2 \tag{11}$$

$$v_2(\hat{x}_{12}) - p_2 = v_1(\hat{x}_{12}) - p_1 \tag{12}$$

$$v_1(\hat{x}_1) - p_1 = 0. \tag{13}$$

If these points are strictly ordered with $0 < \hat{x}_1 < \hat{x}_{12} < \hat{x}_{2B}$, then they represent a complete sorting solution in the vertical differentiated problem. That is, they define a full mixed bundling solution where $\mathcal{N}_B = (1 - F(\hat{x}_{2B}))$ (customers with $x \geq \hat{x}_{2B}$ purchase multi-device access), $\mathcal{N}_2 = (F(\hat{x}_{2B}) - F(\hat{x}_{12}))$, and $\mathcal{N}_1 = (F(\hat{x}_{12}) - F(\hat{x}_1))$. When two or more of the $\hat{x}$'s overlap (e.g., if $\hat{x}_1 = \hat{x}_{12}$) then corresponding X_i sets will be empty, implying zero sales of that menu option. From first-order conditions (corresponding to Eq. 5 and 12 in Bhargava and Choudhary, 2008), the firm's profit under an interior mixed-bundle solution,

$$\Pi = (1 - F(\hat{x}_{2B}))\,(v_B(\hat{x}_{2B}) - v_2(\hat{x}_{2B})) + (1 - F(\hat{x}_{12}))\,(v_2(\hat{x}_{12}) - v_1(\hat{x}_{12})) + (1 - F(\hat{x}_1))\,v_1(\hat{x}_1)$$

is maximized by setting $\hat{x}_1$ as in Eq. 7, $\hat{x}_{12}$ as in Eq. 9 and $\hat{x}_{2B}$ as below.

$$\hat{x}_{2B} = \text{Sol.}\left[\frac{\mathrm{f}(x)}{1 - F(x)} \times \frac{v_B(x) - v_2(x)}{\frac{\partial}{\partial x}\left(v_B(x) - v_2(x)\right)} = 1\right] \tag{14}$$

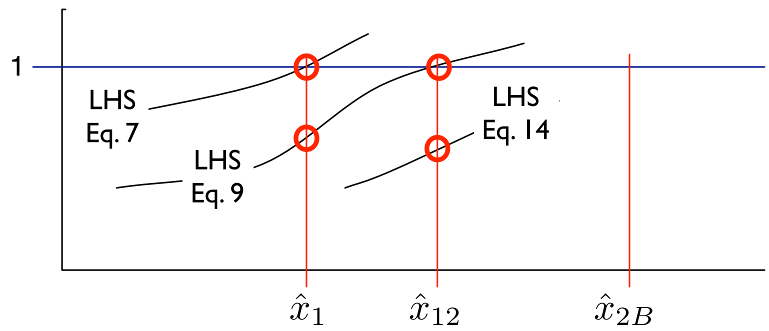


Figure 3: Full mixed bundling is optimal when these critical points $\hat{x}_1, \hat{x}_{12}, \hat{x}_{2B}$ are exactly in the order depicted (equivalently, that the curves are in the order depicted, at the critical points).

The conditions for identification of the indifference points and for full mixed bundling to be optimal are visually illustrated in Fig. 3. If the constraint $\hat{x}_1 < \hat{x}_{12}$ is violated (no one buys device 1 alone), then the partial bundling menu $\{2, B\}$ is plausible, with indifference points $\hat{x}_2$ as in Eq. 8 and $\hat{x}_{2B}$ as in Eq. 14. If $\hat{x}_{12} < \hat{x}_{2B}$ is violated (no one buys device 2 alone), then the menu $\{1, B\}$ is plausible, with indifference points $\hat{x}_1$ (Eq. 7) and $\hat{x}_{1B}$ defined analogously to Eq. 14. When neither of the partial bundles yields strict ordering in the indifference points, then the pure bundle solution is characterized by the indifference point $\hat{x}_B$ defined analogously to Eq. 7. Hence the alternative multi-device selling strategies map to the interior vs. boundary optima to the firm's profit maximization problem.

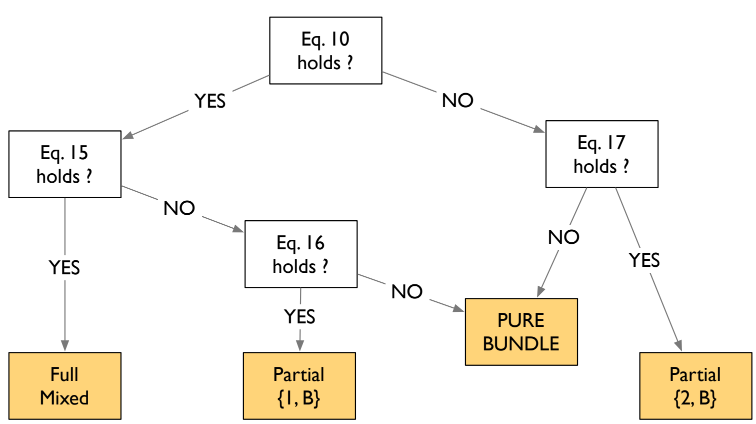


Figure 4: Conditions and logic for the optimality of each multi-device strategy.

Proposition 1. Under Assumption 1 and non-decreasing demand elasticity,

1. full mixed bundling is optimal when $\frac{\partial v_2}{\partial v_1}>0$ at $\hat{x}_1$ (Eq. 10) and

$$\frac{\partial}{\partial x}\left(\frac{v_1(x)-\eta(x)}{v_2(x)-v_1(x)}\right)>0 \quad \text{at } \hat{x}_{12} \tag{15}$$

(relative to having device 2-only, the ratio $\frac{\textit{value gain on adding device 1}}{\textit{value loss on dropping to device 1}}$ *is increasing at* $\hat{x}_{12}$*).*

2. a partial bundle $\{1, B\}$ is better when Eq. 15 fails, but

$$\frac{\partial}{\partial x}\left(\frac{\tilde{v}_2(x)}{v_1(x)}\right)\Big|_{\hat{x}_1}>0 \quad \equiv \quad \frac{\partial}{\partial x}\left(\frac{v_B(x)}{v_1(x)}\right)\Big|_{\hat{x}_1}>0 \tag{16}$$

(propensity for multi-device access (relative to device 1-only) is increasing at $\hat{x}_1$*).*

3. a partial bundle $\{2, B\}$ is better when Eq. 10 fails but

$$\frac{\partial}{\partial x}\left(\frac{\tilde{v}_1(x)}{v_2(x)}\right)\Big|_{\hat{x}_2}>0 \quad \equiv \quad \frac{\partial}{\partial x}\left(\frac{v_B(x)}{v_2(x)}\right)\Big|_{\hat{x}_2}>0 \tag{17}$$

(propensity for multi-device access (relative to device 2-only) is increasing at $\hat{x}_2$*).*

4. pure bundling (single price for accessing both devices) is optimal when Eqs. 16–17 fail, i.e., lower-value consumers have higher propensity for multi-device access.

Fig. 4 graphically depicts the relationship among the conditions and optimal multi-device strategies. While the statements in Proposition 1 provide exact conditions for optimality of the different multi-device strategies (i.e., for $\mathcal{N}_1$ or $\mathcal{N}_2$ to be zero or not), they can also be applied to develop a more practical understanding of when $\mathcal{N}1$ and/or $\mathcal{N}2$ become too small to be viable. When $\mathcal{N}_1$ or $\mathcal{N}_2$ is too small (i.e., the $\hat{x}$ values are too close) then including i-only may not be worth the additional cost of complexity. In the multi-device strategy, it is worthless to sell device 1 by itself when the ratio $\frac{v_1(x)}{v_2(x)}$ is quite similar (or increasing) for all x in the neighborhood of $\hat{x}_1$. That is, not only should the decreasing ratio condition be satisfied, but in fact the ratio should be sharply decreasing in order for the strategy to be truly impactful. Hence the increasing (or decreasing) at $\hat{x}_i$ can be replaced with the more intuitive "increasing (or decreasing) in x." Incidentally, the condition is identical to that needed to ensure positive sales of device 1 in the vertical differentiation benchmark that was developed under the restriction that each consumer purchases at most one device (Eq. 10). Similarly, separate sales of device 2 add no profit when the ratio $\frac{v_1(x)-\eta(x)}{v_2(x)-v_1(x)}$ is quite similar (or decreasing) for all x in the neighborhood of $\hat{x}_{12}$. Notably, these pivotal points $\hat{x}_1$ and $\hat{x}_{12}$ are simply the values that are featured under independent pricing in a standard two-product vertical differentiation problem. Hence the decision problem of picking the best multi-device strategy does not increase the information burden on the firm.

The intuition underlying the findings in Proposition 1 may be obtained by combining theoretical insights from vertical differentiation and bundling theory. First consider the case where Eq. 15 holds, i.e., the variation in bundle disutility leads to an amplification of heterogeneity in consumer preferences. Due to this (and when Eq. 10 holds, see end of §3.5), the firm should offer as vast a product line as possible, using both separate sales and a multi-device discount. When such amplification does not occur, partial bundling can still be optimal; the requirement for that is merely that high-value customers have sufficiently strong preference for the bundle over single-device consumption, thereby reducing pressure from single-device to multi-device price. Finally, pure bundling works best when lower-value consumers have greater propensity for multi-device access (Eqs. 16–17 fail). This is because this kind of variation in bundle disutility helps make the multi-device demand curve flatter: it brings the multi-device valuations of higher-x consumers closer to those of the lower-x consumers. The interplay between the multiple conditions and equations in Proposition 1 may be further understood by considering specific examples of valuation functions.

Corollary 1 (Constant marginal valuations (CMV) for quality)**.** Let $v_2 = \lambda v_1$ and $v_B = \gamma v_1$ (i.e., $\eta(x) = (1+\lambda-\gamma)v_1(x)$), such that $\lambda > 1$, $\gamma \in [\lambda, 1+\lambda]$. Then pure bundling and separate selling of the two devices are equally profitable and produce identical allocation of goods to customers.

Under CMV, all consumers perceive the two products as having the same quality ratio, and both ratios $\frac{\partial \tilde{v}_2(x)}{\partial v_1(x)}$ and $\frac{\partial v_j(x)}{\partial v_i(x)}$ are constant (rather than strictly positive). Eq. 10 fails and Eq. 17 also fails, hence Proposition 1 concludes that selling only the highest quality (pure bundle) is optimal. Corollary 1 covers the setting of Banciu et al. (2010, Proposition 1), who studied bundling of vertically differentiated goods, and also found that pure bundling is optimal under this setting. In fact, the pure bundle solution (under CMV) produces identical allocation and profits as separately selling the two devices (because $\hat{x}_1 = \hat{x}_2 = \hat{x}_{12}$ in Eq. 7–8–10 under CMV) The CMV assumption was also adopted by Calzada and Valletti (2012) in analyzing vertical differentiation in movie distribution (in theaters vs. DVD). The findings for separate sales vs. bundling (under simultaneous distribution) can be obtained from the cost-adjusted case of Proposition 1 or by referring to optimality of price discrimination under positive marginal costs (Anderson and Dana, 2009).

Corollary 2 (CMV with Consumption Tax or other Disutility)**.** Let $v_2(x) = \lambda v_1(x) - \Delta_2$ and $\eta(x) = \Delta_1 + (1+\lambda-\gamma)v_1(x)$, where $\Delta_i > 0$. Then full mixed bundling is optimal when $\Delta_1(\lambda - 1) > \Delta_2(\gamma - \lambda)$, otherwise the optimal strategy is the partial mixed bundle $\{1, B\}$.

Δ_i may be interpreted as a tax, surcharge, disposal fee or some other disutility associated with the superior device (and the even superior bundle). It is identical across all users, hence the burden is disproportionately higher on lower-x consumers. This kind of formulation also applies when there is "no free disposal" of unwanted components of a good (Chellappa and Shivendu, 2010). Then Eq. 10 is satisfied for all $\Delta_2 > 0$ because the ratio $\dfrac{\partial \tilde{v}_i(x)}{\partial v_j(x)}$ is strictly increasing. The condition for *full* mixed bundling (Eq. 15) is satisfied whenever $\Delta_1(\lambda - 1) > \Delta_2(\gamma - \lambda)$, e.g., when device 2 is highly superior to device 1 (high λ). Otherwise, Eq. 16 holds and the optimal bundling strategy is a partial bundle $\{1, B\}$. Intuitively, when higher-valuation customers "care sufficiently more" for multi-device access (i.e., they have a higher proportional increase in value relative to single-device access), then the firm can profitably segment the market. When devices 1 and 2 are relatively similar, then the optimal strategy is to sell the bundle and device 1-alone, but not device 2-alone; doing so would increase competition between the products and reduce the margin for the multi-device bundle. When device 2 is vastly superior, then the margin concern is reduced (even though competition occurs) because device 2-alone can also command a high price. In this case, the optimal strategy is to sell the full mixed bundle.

It is hard to apply Proposition 1 definitively to specific products without accurate empirical data about users' demand preferences. However, as an example of its relevance, consider the following conjecture with regard to high-end software such as Autodesk's products for 3D design in engineering and entertainment. Autodesk products were traditionally available for computer workstations but now are also available for mobile devices. These products are used by both industry professionals and mass consumers such as amateurs, students, and hobbyists. Not only do professionals have higher value for computer-based access, but likely would experience a big drop in valuation if instead they switched to smartphone or tablet access which provides a relatively crippled user experience for such complex applications. Mass consumers, on the other hand, have a smaller relative drop in valuation on shifting from workstation to tablet version. Generalizing this comparison, the ratio $\frac{v_2(x)}{v_1(x)}$ is increasing in x, and Eq. 10 is satisfied, implying that some bundling is desirable. Next, to examine the type of bundling, consider the relative propensity for multi-device access. It is likely that professionals have higher need for multi-device access because they might use the workstation access for design, and tablet version for communication with clients. Hence $\eta(x)$ is increasing in x. If the increase in sharp enough to satisfy Eq. 15 then full mixed bundling is optimal. If it is not, then Eq. 16 is likely to be satisfied (because of the observations regarding how $v_1(x)$ varies with x), making the partial bundle $\{1, B\}$ the optimal strategy. The firm should offer a relatively low-priced access over mobile devices, and bundle mobile access for free for buyers of a premium-priced workstation product. Of course, in practice, it might be useful to add a limited-feature workstation version to mid-range consumers.

A few notable aspects about Proposition 1. First, its guidance about the design of a multi-device strategy requires little additional information than what the firm might need to sell the devices separately without a coordinated bundle discount. The firm must know valuations around the critical points $\hat{x}_1$ and $\hat{x}_2$, but this is needed even under a single-device strategy with either device 1 or 2 respectively. And knowledge about contingent valuations $\hat{v}$ is also needed even if the firm were selling both devices with independent pricing (no bundle discount). The only additional information needed is whether the various ratios in the proposition are increasing or decreasing in x especially at these critical points. Second, the result holds regardless of the specific distribution of valuations as long as the demand functions have non-decreasing demand elasticities; this requirement is satisfied by distributions that have a monotone hazard rate. Third, the result does not require consumers to have either the same or proportional disutility, and does not require specific functional relationships between valuations of each device. In fact, Corollary 1 underlies the need for caution when developing and interpreting results from highly specific formulations such as linear utility or constant marginal valuations.

4.2 Independent devices

Consider the case where the two devices are essentially independent in value-provision, so that a consumer's valuation for one device provides no additional information about her value for the second device (i.e., the distributions f_1 and f_2 have zero correlation). Then separate pricing (denoted by the superscript ^{S}) is optimal, and there is no need for multi-device discounting. As noted earlier, the two devices may be unequal in terms of total value under the demand curve, with device 2 (labeled as the existing technology) having the superior demand profile. For mathematical and computational convenience in analyzing this case, it is useful to employ a specific distribution of standalone valuations for each device. In absence of any other information, we assume that the marginal distributions f_1 and f_2 have uniform density.

Assumption 2 (Demand for each device)**.** Consumer valuations for obtaining service via device 1 (or 2) are distributed uniformly in an interval $[0, a_1]$ (respectively, $[0, a_2]$).

Assumption 3 (Zero Correlation)**.** A consumer's valuations for devices i and j are independent. The joint distribution covers the rectangle $[(0,0),(0,a_2),(a_1,a_2),(a_1,0)]$.

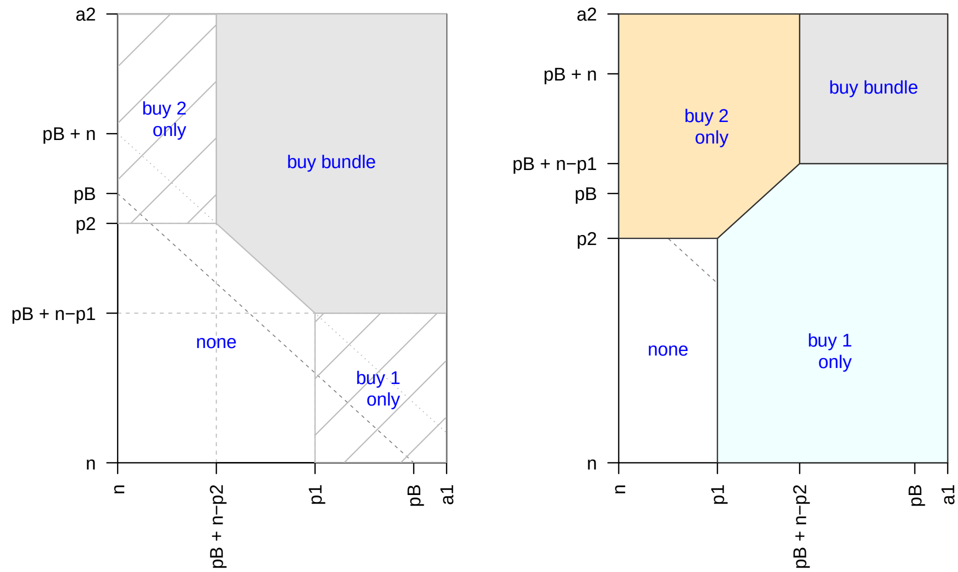
With these assumptions, cross-price effects on device sales are identical ($\frac{\partial(\mathcal{N}_j+\mathcal{N}_B)}{\partial p_i} = \frac{\partial(\mathcal{N}_i+\mathcal{N}_B)}{\partial p_j}$), because both changes correspond to the same margin $(v_i - p_i) = (v_j - p_j)$. In the base case of independent pricing ($\delta = 0$), the firm maximizes its profit $\Pi^S = (p_1 - c1)(\mathcal{N}_1 + \mathcal{N}_B) + (p_2 - c_2)(\mathcal{N}_2 + \mathcal{N}_B)$. Some customers will purchase just a single device while others will be multi-device buyers. Even when the firm offers no bundle discount, it cannot ignore the effects of dual consumption disutility in setting its prices p_1, p_2. Now, set $q_i(p_1, p_2) = \mathcal{N}_i(p_1, p_2) + \mathcal{N}_B(p_1, p_2)$, and consider the impact of price adjustments. Differentiating Π^S by p_i (for $i = 1, 2$) and employing the property $\frac{\partial q_2}{\partial p_1} = \frac{\partial q_1}{\partial p_2}$, the optimal independent prices p_i^S (for $i = 1, 2$) can be written as

$$p_i^S = \text{Solution of } \left(-\frac{t}{q_i} \frac{\mathrm{d}q_i(c_1 + t(p_1 - c_1), c_2 + t(p_2 - c_2))}{\mathrm{d}t} = 1 \right) \Bigg|_{t=1}. \tag{18}$$

This condition parallels the standard first-order condition for price optimization for a single good, $\frac{\partial D(p)}{\partial p} \frac{p-c}{D(p)} = 1$, which is "elasticity = inverse percentage markup" (seen by rewriting it as $\frac{\partial D(p)}{\partial p} \frac{p}{D(p)} = \frac{p}{p-c}$). Eq. 18 essentially states that at optimal independent prices, a 1% increase in markup induces a 1% fall in demand for each device (Armstrong, 2013). Now, applying Lemma 1 at (p_1^S, p_2^S) yields a sufficient condition for optimality of offering a bundle discount: *elasticity of bundle demand at* (p_1^S, p_2^S) *must exceed elasticity of aggregate single-device demand at those prices.* The insight from Lemma 1 can be refined by giving some structure to the disutility term. One possibility is that all consumers experience identical disutility η, with single-device valuations distributed over $[\eta, a_i]$ rather than $[0, a_i]$ to ensure free disposal (Armstrong, 2013).

Assumption 4 (Constant Disutility of Dual Consumption)**.** All consumers have identical disutility η, and $v_B = v_1 + v_2 - \eta$.

A constant disutility of dual consumption acts as a dual-access surcharge on all consumers, or an extra cost passed from the firm to consumers. Consumers perceive a price increase η when pursuing dual consumption while the firm gains no additional revenue. Then, market shares for each option can be specified as shown in Fig. 5. When $\delta \geq \eta$ (panel (a) in the figure), then Eq. 1 can be applied, replacing p_B with $p_B + \delta$. The other case $\delta \leq \eta$ (panel b) behaves like independent pricing (as if η was reduced by δ because $p_B < (p_1 + p_2)$), and



(a) High discount: $\delta \geq \eta$.
(b) Low discount: $\delta \leq \eta$.

Figure 5: Product sales under prices p_1, p_2, p_B. Each point (v_1, v_2) in the rectangle represents a consumer's standalone valuations for the pair of devices. The regions in both panels work in all the possible cases $p_B \leq a_1 \leq a_2$; $a_1 \leq p_B \leq a_2$; and $a_1 \leq a_2 \leq p_B$.

consumers in the top-right corner purchase both devices. Precise formulæ for $\mathcal{N}_i, \mathcal{N}_B$ under Assumptions 2-3-4 are given below.

	low bundle discount$(\delta \leq \eta)$	**high bundle discount**$(\delta \geq \eta)$
$\mathcal{N}_1:$	$(a_1 - p_1)(p_2 + \eta - \delta) - \dfrac{(\eta-\delta)^2}{2};$	$(a_1 - p_1)(p_2 + \eta - \delta).$
$\mathcal{N}_2:$	$(a_2 - p_2)(p_1 + \eta - \delta) - \dfrac{(\eta-\delta)^2}{2};$	$(a_2 - p_2)(p_1 + \eta - \delta).$
$\mathcal{N}_B:$	$(a_1 - p_1 - \eta + \delta)(a_2 - p_2 - \eta + \delta);$	$(a_1 - p_1 - \eta + \delta)(a_2 - p_2 - \eta + \delta) - \dfrac{(\eta-\delta)^2}{2}.$

To test for when a multi-device strategy with bundle discount is attractive, first consider independent pricing (no discount, $\delta = 0 < \eta$). Now, $q_i = \mathcal{N}_i + \mathcal{N}_B = a_j(a_i - p_i - \eta) + \eta p_j + \frac{\eta^2}{2}$. The profit function (with zero marginal costs and Assumptions 2-3-4) is $\Pi^S = p_1 q_1 + p_2 q_2$.

Taking derivatives with respect to p_1 and p_2, optimal independent prices are

$$p_1^S = \frac{a_1}{2}\left(1 + \frac{\eta^2}{2(a_1 a_2 - \eta(a_1 + a_2))}\right); \qquad p_2^S = \frac{a_2}{2}\left(1 + \frac{\eta^2}{2(a_1 a_2 - \eta(a_1 + a_2))}\right). \tag{19}$$

Plugging these prices into $\frac{\partial \Pi}{\partial \delta}$ at $\delta = 0$ (Eq. 5) generates the condition for a bundle discount to be profitable.

$$\begin{aligned} 4\left(\frac{a_1}{\eta}\frac{a_2}{\eta}\right)^3 + 4\left(\frac{a_1}{\eta} + \frac{a_2}{\eta}\right)^2\left(\frac{a_1}{\eta}\frac{a_2}{\eta} - 2\right) + 20\frac{a_1}{\eta}\frac{a_2}{\eta}\left(\frac{a_1}{\eta} + \frac{a_2}{\eta}\right) \\ > \left(\frac{a_1}{\eta}\frac{a_2}{\eta}\right)^2\left[12 + 8\left(\frac{a_1}{\eta} + \frac{a_2}{\eta}\right)\right] + 3\left(\frac{a_1}{\eta}\frac{a_2}{\eta}\right) \end{aligned} \tag{20}$$

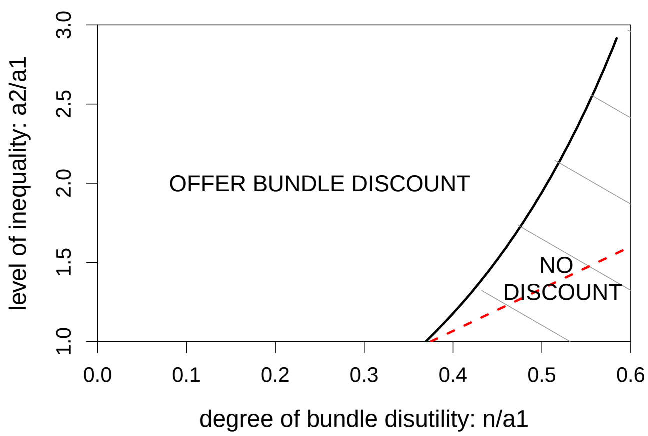


Figure 6: Disutility threshold for which offering a bundle discount is profitable. The dashed line represents a percentage increase in inequality equal to percentage increase in disutility.

Fig. 6 plots the condition and the threshold for employing a multi-device strategy (either full or partial mixed bundling). Bundling is optimal when there is no penalty and the devices have demand profiles of equal strength. As disutility increases, the gain from bundle discounting decreases and becomes zero when the penalty hits the threshold given by Eq. 20. Thus, multi-device discounting is less useful when disutility (normalized against a_1) is high. It also appears that a higher disutility threshold is acceptable as the level of inequality increases; however, as the dashed line in the figure indicates, the increase in threshold is less than a proportional increase. That is (starting at the threshold for symmetric demand profiles), the threshold penalty level rises at a slower rate than the increase in penalty. Equivalently, a $k\%$ increase in a_2 accompanied by a $k\%$ increase in disutility will make bundle discounting unprofitable. These findings are summarized below.

Proposition 2. Under Assumptions 2-3-4 for devices with demand profiles of similar strength, multi-device discounting increases profit when disutility of dual consumption is low, as in Eq. 20.

Armstrong (2013, Proposition 2) provides a *sufficient* (but not necessary) condition for profitability of bundle discount when the goods are symmetric (have identical demand distributions, either independent or non-independent). Adapting that result to our notation and the setting of independent and uniform single-device demand, the condition is that the ratio $\frac{N_B}{\mathcal{N}_1}$ be decreasing for all p. However, this condition is satisfied only at very small values of η close to 0; this may be seen by computing the ratio (without loss of generality) at $a_1 = a_2 = 1$. Hence the application of that result merely confirms the known result that bundling is optimal in the case of additive valuations. In contrast, Eq. 20 produces the tight threshold level of disutility ($\eta = 0.369$ relative to $a_1 = 1$) for which multi-device discounting is still profitable.

Proposition 2 and Eq. 20 establish *when* multi-device bundling is attractive. Now consider the *design* of the bundling strategy: should the firm practice full mixed bundling, or is partial bundling better? This question is answered by evaluating the boundary solutions $p_i = p_B$ for $i = 1, 2$ for the "high discount" case (the low discount case is eliminated because it is trivially inconsistent with this boundary solution). The firm's optimization problem under high discount is

$$\begin{aligned}\underset{p_1,p_2,\tilde{p}_B}{\text{Max.}}\ \Pi = & \ p_1(a_1 - p_1)(p_B - p_1) + p_2(a_2 - p_2)(p_B - p_2) \\ & + p_B\left((a_2 - \tilde{p}_B + p_1)(a_1 - \tilde{p}_B + p_2) - \frac{(p_1 + p_2 - \tilde{p}_B)^2}{2}\right) \qquad (21) \\ \text{s.t.}\quad & 0 \le p_1 \le \min\{a_1, p_B\}, \quad 0 \le p_2 \le \min\{a_2, p_B\}, \quad \tilde{p}_B \le p_1 + p_2 + \eta.\end{aligned}$$

Lemma 2 (Single-device Prices)**.** Under Assumptions 2-3-4, optimal single-device prices for separate sales are $p_1 = \frac{2a_1}{3}$ and $p_2 = \frac{2a_2}{3}$; otherwise, $p_i = p_B$.

The boundary $p_i = p_B$ in the above problem represents partial bundling. To evaluate the optimal design under partial bundle, suppose device 1 is offered separately in addition to the multi-device bundle. The firm's profit function is obtained by plugging $p_2 = p_B = p$ into Eq. 21. Optimization yields $p_1^* = \frac{2a_1}{3}$ and $p^* = \frac{4a_1^2+6a_1a_2-6a_1\eta-6a_2\eta+3\eta^2}{a_1-\eta}$. To examine when this partial bundle design is optimal (i.e., beats full mixed bundling), consider the firm's full bundling strategy, computing the profit term Π^ϵ in Eq. 21 by replacing p_2 with $p^*-\epsilon$ and p_B with p^*. Computing the derivative $\frac{\partial \Pi^\epsilon}{\partial \epsilon}$ at $\epsilon = 0$ yields the condition for partial vs. full mixed bundling.

Proposition 3. When consumer valuations for the traditional and emerging devices are mutually independent with demand profiles satisfying Assumptions 2-3-4, then the optimal strategy is

1. **full mixed bundling**: sell separate access to each device as well as discounted access to the bundle, when the single-device demand profiles are not too dissimilar, as in

$$a_1 \in \left(\frac{1}{4}a_2 + 3\eta - \sqrt{a_2^2 - 2a_2\eta - 3\eta^2}, a_2\right) \approx \left(\frac{a}{2} + \frac{\eta}{2}, a_2\right). \tag{22}$$

2. **partial bundling**: provide buyers of the traditional (stronger) device with free access to the emerging (weaker) device, and offer separate access to the emerging device, when the emerging device is sufficiently weak.

A high disutility for dual consumption makes it more likely that full mixed bundling outperforms partial bundling.

A full mixed bundling strategy is optimal when the two devices have comparable aggregate distribution of valuations; otherwise, a partial bundling strategy is optimal wherein the device which commands far stronger valuations is not offered separately. Consider the *Wall Street Journal*'s approach for managing online and print access. The print product—which is the traditional device for consuming the newspaper, and presumably still the one with a stronger demand profile—now includes free access to the digital edition, while the latter is also offered separately; but there is no print-only product sold at a lower price than the bundle. While the partial bundling strategy resembles "tying" it does not have the typical motive behind tying, which is to leverage a strong product into driving sales of the weaker product or one that has strong competition. Instead the main reason to avoid selling device 2 by itself is (following the maximal differentiation principle) to avoid internal competition between two options that have relatively similar value.

How does bundle disutility impact the firm's choice of partial vs. full mixed bundling? When $\eta = 0$ (and under an identical model setting with zero marginal costs), Bhargava (2013) showed that the threshold for the design of bundling strategy is $a_1 = \frac{a_2}{2}$: partial bundling becomes active when a_1 is below $\frac{a_2}{2}$. Since the threshold is now increased to $\approx \frac{a_2+\eta}{2}$, Proposition 3 provides the insight that a positive disutility makes full mixed bundling *more* likely. This is because full bundling actually does more to promote single-device sales than partial bundling (because the latter eliminates separate sales of device 2). When disutility is high, the firm has less incentive to eliminate separate sales of device 2.

The condition of independent and additive valuations applies to, or approximates, several applications of multi-device access. For instance, consider two devices—workstation and tablet—for an enterprise information system that lets users view informational reports as well as manipulate the data in the system. Workstation access to the system might be needed for making full use of multimedia and graphical report formats or for entering large amounts of data. Tablet device access might provide mobile access and be highly convenient for browsing predefined reports and minimal data entry. The tablet device provides *added* functionality (mobility, convenience) rather than substituting the value provided by workstation access. The zero correlation assumption may also fit this setting because users' valuations for workstation access provide little information about their value for mobility and convenience. Proposition 3 then provides clear guidance for designing the multi-device strategy. Prior work by Long (1984) and McAfee et al. (1989) suggests that the design guidelines provided in Proposition 3 are actually robust and applicable even when valuations are sub-additive and when the distributions of consumer valuations are not independent (as long as they are not too positively correlated).

4.3 Negative correlation: Horizontally Differentiated devices

Now consider the case where the two devices are horizontally differentiated. This may occur when they differ sharply on certain attributes for which the consumer base has divergent preferences. For instance, certain consumers highly value portability and convenience, and are willing to consume videos and other content on tiny smartphone screens. Other consumers disdain such consumption and have high value for consuming the content on large fixed television sets. In this setting, high (conversely, low) value for one device is a signal of low (respectively, high) valuation on the other device. This tension between devices is apparently occurring presently with respect to TV and movie content. The traditional model was bundles of channels accessed at home, and on a schedule, via cable or satellite service providers such as Comcast or Dish Network. The new model is access anywhere, anytime, over the Internet. The traditional model appears to be popular with "older Americans" while "the under-40 set ... just want their video content when and where they want it" (Stelter, 2013).

The case of dichotomous device preferences is nicely captured by adopting the following formulation. Let device-1 valuations be $v_1(x)=ax\in[0,a]$ and device-2 valuations be $v_2(x)=\frac{a_2}{a_1}(a_1 - v_1(x)) = \theta a(1-x) \in [0, \theta a]$ (with $\theta \geq 1$). The index of consumer types x has a cumulative distribution function F on $[0,1]$. The stronger a consumer's preference for one device, the weaker her preference for the other device. To allow various levels of disutility in this formulation, write $\eta(x)=\frac{v_1(x)v_2(x)}{v_1(x)+v_2(x)}=\frac{\theta ax(1-x)}{x+\theta(1-x)}$. Then, contingent valuations for access via each device (contingent on already having the other) are $\tilde{v}_1(x)=\frac{ax^2}{x+\theta(1-x)}$, and $\tilde{v}_2(x)=\frac{a(\theta x)^2}{x+\theta(1-x)}$. To keep the exposition concrete and simple, I will assume that F is a uniform distribution. This assumption has little loss of generality, because for any other distribution of x we can reparameterize the v_1 and v_2 valuations so that F becomes a uniform distribution.

With this setting, optimal single-device prices in the absence of a multi-device strategy (i.e., separate sales, no bundle discount) are easily computed. First, note that optimal prices under separate selling with no disutility of dual consumption—i.e., $p_1 = \frac{a}{2}$ and $p_2 = \frac{\theta a}{2}$—are unaffected even when there is a positive bundle disutility. This is because the two devices cover the market but have non-overlapping sales at these prices. There is no demand for joint consumption, hence consumers feel no impact of that disutility. Similarly, the firm sees no merit in reducing either price (to encourage dual consumption) because the price times volume relationship for each device is already optimized. Hence $p_1 = \frac{a}{2}$ and $p_2 = \frac{\theta a}{2}$ are optimal even under bundle disutility. Second, if the device demand profiles had the same strength (i.e., $\theta = 1$) then a pure bundle solution (with price $\frac{a}{2}$) would produce identical results as separate selling. If the demand profiles were unequal, then pure bundling could only do worse (for the same reason, that price times volume is already optimized, hence increasing sales has no positive impact on revenue).

Next, consider mixed bundling, and the impact of a small bundle discount δ. Note that at the separate selling prices, the marginal customer at $\tilde{x} = \frac{1}{2}$ obtains zero surplus from either device. Without loss of generality, assign this customer to device 2, and consider what it would take to entice such a customer to also purchase device 1. Computing her contingent valuation for device 1 ($\tilde{v}_1(\frac{1}{2}) = \frac{a}{2}\frac{1}{1+\theta}$) shows that the minimum discount needed would be $\delta = p_1 - \tilde{v}_1(\frac{1}{2}) = \frac{a}{2}\frac{\theta}{1+\theta}$, hence the firm must set a bundle price just below $p_B = \frac{a}{2}\left(1+\theta-\frac{1}{1+\theta}\right)$. But since this price exceeds both p_1 and p_2, this approach trivially enhances the firm's profit, because no customer generated as much revenue under separate selling.[6] This establishes the unsurprising result that practicing some form of multi-device discount is optimal, regardless of the relative strength of demand profiles of the individual devices. It is also evident that the higher the inequality in profiles (higher θ), the greater the bundle discount needed for the multi-device strategy to make an impact, making the bundling strategy less attractive.

Finally, consider the design of the multi-device strategy, and whether the firm should

[6]The parallel argument does not work if device 1 is assigned to the marginal consumer because the minimal discount required for multi-device consumption makes p_B below p_2, hence partial bundling $\{2, B\}$ is never optimal.

sell separate access under both or just one device. A partial bundle strategy–say $\{1, B\}$—requires that *all* single-device buyers of device 2 become bundle buyers. But, since consumers with the highest-value for device 2 also have the lowest value for device 1 (zero, under our formulation), achieving the partial bundle outcome would require a very deep bundle discount ($p_B = p_2$ in our setting), thus generating no more revenue than the full mixed bundle solution. More generally, if the negative correlation in valuations were not so extreme, even then a partial bundle solution would be relatively less attractive; positive marginal costs would further diminish the case for partial bundling. Putting everything together, the insights from this stylized formulation can be stated as follows.

Proposition 4. A full mixed bundling strategy is optimal when device demands are negatively correlated. A discounted bundle price entices customers with moderate preferences to purchase multi-device access, and enables the firm to increase single-device prices, targeted to the higher- and extreme-value consumers. Multi-device discounting is more attractive under low disutility, and when there is overlapping demand under separate single-device prices.

5 Conclusion

The distribution and consumption of digital products is undergoing a fundamental transformation towards multi-device access. Content providers are experimenting with multiple models for provision and pricing of multi-device consumption. This paper is the first to formally analyze and provide insights regarding multi-device strategies. The problem of designing a multi-device strategy is framed in terms of bundling access under two related devices, creating a choice between pure bundling (one price gets both devices), mixed bundling (price each device separately, and offer discount for getting both) and partial bundling (one device is sold separately and is also available bundled into the second). Mathematical models for bundling are inherently complex to analyze, and the relevant model for this paper has multiple complexities due to correlated demand distributions, sub-additive valuations, and devices with unequal demand profiles. Besides tackling the multi-device strategy question, this paper makes theoretical contributions towards the modeling and analysis of product bundling by developing an analytical modeling framework that covers these complexities, is tractable, and produces useful insights regarding alternative bundling strategies. By decomposing the problem into parts, this framework produces results that are more precise and actionable than the properties that emerge from a completely general analysis of the economics of bundling (e.g., Armstrong (2013)), and more robust and general than numerical or application-based findings (e.g., Venkatesh and Chatterjee (2006); Kannan et al. (2009)).

The practical insights from this analysis are summarized in Table 1. When the devices are vertically differentiated and there is positive correlation in consumers' valuations for the two devices, then some form of multi-device strategy is always attractive over separate selling (except for the degenerate case where consumers have identical valuations for the two devices). The optimal form is pure bundling when lower-value consumers have greater propensity for multi-device access, because the varying disutility of dual consumption helps make the multi-device demand curve flatter. Otherwise, when bundle disutility amplifies the heterogeneity in consumer preferences, the firm should promote some single-device sales through either a full mixed bundle or a partial mixed bundling strategy. When consumer valuations for the traditional and emerging devices are mutually independent with demand profiles, then full mixed bundling is optimal if the demand profiles for the two devices are relatively similar. Partial bundling is the better design when single-device demand profiles are more unequal; the superior device should include access to the weaker one, while the latter is also sold separately. When devices behave more like substitutes and consumer valuations are negatively correlated, then such partial bundling is less desirable and full mixed bundling is optimal. With all of these results, it is useful to note that identification of the best multi-device strategy requires essentially the same information needed to design a separate selling strategy with independent single-device prices.

While multi-device access is relevant to a wide spectrum of information goods, the present paper and model is best suited to digital *content* goods (such as newspapers and magazines, movies, music, video games). It can also be applied on *computing* goods such as software, and such application motivates several extensions to this research. These include accounting for repeat consumption over time (software, unlike a movie, is not consumed in a single session, and the same code may be executed hundreds of times), and the impact of direct network effects (which are present and strong for many software goods). While these are worthy extensions, they are best left out in a first-round model. The applicability of the current model is limited when products (such as TV shows) are distributed through intermediaries who aggregate the product into a bundle of products from other competing producers. For such products, the economic distortions caused by cross-producer aggregation must be taken into account (see e.g., Bhargava (2012)); again, these are not considered in the present model but would be a good future extension. Additional ideas for future research include addressing the case when only a fraction of the market has the capability for multi-device access, as well as strategy dynamics due to a shift in consumer preferences for different devices over time.

A Appendix

Proof of Proposition 1. First we derive conditions for pure bundling, i.e., offering *only* the firm's "highest quality" product which is multi-device access. Applying Lemma 3 in Bhargava and Choudhary (2008), selling bundled multi-device access only (i.e., only selling the most superior product quality) is optimal when it is superior to each partial bundling strategy, $\{1, B\}$ and $\{2, B\}$. From Proposition 2 in the same paper, these conditions are, respectively, $\frac{\partial}{\partial x}\left(\frac{v_1(x)}{v_B(x)}\right) \geq 0$ and $\frac{\partial}{\partial x}\left(\frac{v_2(x)}{v_B(x)}\right) \geq 0$. Inverting the ratios, the conditions are $\frac{\partial}{\partial x}\left(\frac{v_B(x)}{v_1(x)}\right) \leq 0$ and $\frac{\partial}{\partial x}\left(\frac{v_B(x)}{v_2(x)}\right) \leq 0$. Finally, writing v_B as $v_i + \tilde{v}_j$ yields the conditions as $\frac{\partial}{\partial x}\left(\frac{\tilde{v}_2(x)}{v_1(x)}\right) \leq 0$ and $\frac{\partial}{\partial x}\left(\frac{\tilde{v}_1(x)}{v_2(x)}\right) \leq 0$ respectively. The converses of these conditions yield the cases where partial bundling beats pure bundling.

Next, consider the conditions for full mixed bundling to be optimal (see Fig. 3). Now, because of non-decreasing demand elasticities, all LHS terms in Eq. 7-9-14 are weakly increasing and thus each have a unique intersection with the constant horizontal line at 1. The ordering constraint $0<\hat{x}_1<\hat{x}_{12}<\hat{x}_{2B}$ is satisfied if the LHS terms (after dropping the identical component $\frac{\mathrm{f}(x)}{1-F(x)}$) have a descending order, at least at $\hat{x}_{12}$. Comparing Eq. 7 and 9, and inverting the condition yields the first requirement for mixed bundling (Eq. 10). Comparing Eq. 9 and 14, the second and final requirement is that $\frac{\mathrm{d}}{\mathrm{d}x}\frac{v_2(x)-v_1(x)}{v_B(x)-v_2(x)}<0$ at $\hat{x}_{12}$. Invert this and write v_B-v_2 as $v_1-\eta$ to get Eq. 15. Note the equivalence between the sign of the derivatives of the following terms: $\frac{\tilde{v}_1(x)}{\tilde{v}_2(x)}$; $\frac{\tilde{v}_1(x)}{v_2(x)-v_1(x)}$; $\frac{\tilde{v}_2(x)}{v_2(x)-v_1(x)}$. ■

Proof of Lemma 2. Differentiating Eq. 21 with respect to p_1 and p_2 we get the two first-order conditions (corresponding to $i=1,2$) reduce to $(2a_i-3p_i)(p_B-p_i)=0$. Hence the single-device price $\frac{2a_i}{3}$ is valid so long as p_B exceeds this amount. When equality is attained, there is no longer any taker for the single-device price, and a partial bundling solution is obtained. ■

Proof of Proposition 3. For the partial bundle solution, standard optimization procedure yields $p_1^*=\frac{2a_1}{3}$ and

$$p_2^*=p_B^*=p^*=\frac{4a_1^2+6a_1a_2-6a_1\eta-6a_2\eta+3\eta^2}{a_1-\eta}.$$

Plug this into Eq. 21, replacing p_2 with $p^*-\epsilon$ and p_B with p^*. For full mixed bundling to be optimal, profit should increase as ϵ increases at $\epsilon=0$. This term is

$$\left.\frac{\partial\Pi^\epsilon}{\partial\epsilon}\right|_{\epsilon=0}>0\equiv\left[a_1\left(\frac{a_1}{a_1-\eta}+3\right)-2a_2-3\eta\right]>0,$$

hence full mixed bundling is optimal when $a_1>\frac{1}{4}\left(a_2+3\eta+\sqrt{a_2^2-2a_2\eta-3\eta^2}\right)$ (the negative sign of the square root is rejected because it produces a contradiction at $\eta=0$). Now, $a_2+3\eta+\sqrt{a_2^2-2a_2\eta-3\eta^2}$ has its minimum at $2a_2$ (with η restricted to $\frac{a_2}{3}$). Hence the threshold for mixed bundling always exceeds $\frac{a_2}{2}$, which is the threshold in absence of disutility, implying that disutility makes full mixed bundling less likely. ■

References

Adams, William James and Janet L. Yellen (1976). "Commodity Bundling and the Burden of Monopoly". In: *The Quarterly Journal of Economics* 90.3, pp. 475–498. URL: http://ideas.repec.org/a/tpr/qjecon/v90y1976i3p475-98.html.

Anderson, Eric T. and James D. Dana (2009). "When Is Price Discrimination Profitable?" In: *Management Science* 55.6, pp. 980–989. ISSN: 0025-1909. DOI: http://dx.doi.org/10.1287/mnsc.1080.0979.

Armstrong, Mark (2013). "A more general theory of commodity bundling". In: *Journal of Economic Theory* 148, pp. 448–472.

Bakos, Yannis and Erik Brynjolfsson (1999). "Bundling information goods: Pricing, profits, and efficiency". In: *Management Science* 45.12, pp. 1613–1630.

Banciu, Mihai et al. (2010). "Bundling Strategies When Products Are Vertically Differentiated and Capacities Are Limited". In: *Management Science* 56.12, pp. 2207–2223.

Bhargava, Hemant K. (2012). "Retailer-Driven Product Bundling in a Distribution Channel". In: *Marketing Science* 31.6, pp. 1014–1021.

— (2013). "Mixed Bundling of Two Independently-Valued Goods". In: *Management Science*.

Bhargava, Hemant K. and Vidyanand Choudhary (2008). "When is Versioning Optimal for Information Goods?" In: *Management Science* 54 (May 2008), pp. 1029–1035.

Burstein, M. L. (1960). "The Economics of Tie-In Sales". In: *The Review of Economics and Statistics* 42 (1 1960), pp. 68–73.

Calzada, Joan and Tommaso Valletti (2012). "Intertemporal Movie Distribution: Versioning When Customers Can Buy Both Versions". In: *Marketing Science* 31.4, pp. 649–667.

Carbajo, José et al. (1990). "A Strategic Motivation for Commodity Bundling". In: *The Journal of Industrial Economics* 38.3.

Chellappa, Ramnath K. and Shivendu Shivendu (2010). "Mechanism Design for "Free" but "No Free Disposal" Services: The Economics of Personalization Under Privacy Concerns". In: *Management Science* 56.10, pp. 1766–1780. ISSN: 0025-1909.

Chen, Yongmin and Michael H. Riordan (2013). "Profitability of Product Bundling". In: *International Economic Review* 54.1, pp. 35–57. ISSN: 1468-2354. DOI: 10.1111/j.1468-2354.2012.00725.x. URL: http://dx.doi.org/10.1111/j.1468-2354.2012.00725.x.

Eisenmann, Thomas et al. (2011). "Platform envelopment". In: *Strategic Management Journal* 32.12, pp. 1270–1285. DOI: 10.1002/smj.935.

Evans, David S. and Michael A. Salinger (2005). "Why Do Firms Bundle and Tie? Evidence from Competitive Markets and Implications for Tying Law". In: *Yale Journal on Regulation* 22.1.

Fang, Hanming and Peter Norman (2006). "To bundle or not to bundle". In: *The RAND Journal of Economics* 37.4, pp. 946–963. ISSN: 1756-2171. DOI: 10.1111/j.1756-2171.2006.tb00065.x. URL: http://dx.doi.org/10.1111/j.1756-2171.2006.tb00065.x.

Hagey, Keach (2013). "Magazines Cross The Digital Divide". In: *The Wall Street Journal* January 19, B1. URL: http://online.wsj.com/article/SB10001424127887323706704578227880541302630.html.

Ibragimov, Rustam (2005). "Optimal Bundling Strategies for Complements and Substitutes with Heavy-Tailed Distributions". Harvard Institute of Economic Research Working Paper #2008.

Ives, Nat (2012). "Digital Subs Rising, The Economist Unbundles Tablet Editions From Print". In: *Ad Age* November 30. URL: http://adage.com/article/media/economist-unbundles-tablet-editions-print-subs/238566/.

Kannan, P. K. et al. (2009). "Practice Prize Winner - Pricing Digital Content Product Lines: A Model and Application for the National Academies Press." In: *Marketing Science* 28.4, pp. 620–636.

Kobayashi, Bruce H. (2005). "Does Economics Provide a Reliable Guide to Regulating Commodity Bundling by Firms? a Survey of the Economic Literature". In: *Journal of Competition Law and Economics* 1.4, pp. 707–746. DOI: 10.1093/joclec/nhi023. eprint: http://jcle.oxfordjournals.org/content/1/4/707.full.pdf+html. URL: http://jcle.oxfordjournals.org/content/1/4/707.abstract.

Koukova, Nevena T. et al. (2008). "Product form bundling: Implications for Marketing Digital Products". In: *Journal of Retailing* 84 (2 2008), pp. 181–195.

Lewbel, Arthur (1985). "Bundling of substitutes or complements". In: *International Journal of Industrial Organization* 3.1, pp. 101–107.

Long, J. (1984). "Comments on "Gaussian Demand and Commodity Bundling"". In: *Journal of Business* 57.1, S235–S246.

Matutes, Carmen and Pierre Regibeau (1992). "Compatibility and Bundling of Complementary Goods in a Duopoly". In: *The Journal of Industrial Economics* 40.1, pp. 37–54.

McAfee, R. Preston et al. (1989). "Multiproduct Monopoly, Commodity Bundling, and Correlation of Values". In: *The Quarterly Journal of Economics* 104.2, pp. 371–83.

Prasad, Ashutosh et al. (2010). "Optimal Bundling of Technological Products with Network Externality". In: *Management Science* 56.12, pp. 2224–2236.

Salinger, Michael A. (1995). "A Graphical Analysis of Bundling". In: *The Journal of Business* 68.1, pp. 85–98.

Schmalensee, Richard (1984). "Gaussian Demand and Commodity Bundling". In: *The Journal of Business* 57.1, S211–230.

Stelter, Brian (2013). "Sony and Viacom Reach Tentative Deal to Stream Cable Channels". In: *The New York Times* August 15.

Stigler, George (1963). "United States v. Loew's Inc: A Note on Block-Booking". In: *The Supreme Court Review*, pp. 152–157.

Stremersch, Stefan and Gerard J. Tellis (2002). "Strategic Bundling of Products and Prices: A New Synthesis for Marketing". In: *Journal of Marketing* 66.1, pp. 55–72.

Suroweicki, James (2010). "Bundles of Cable". In: *The New Yorker* (25 2010).

Venkatesh, R. and Rabikar Chatterjee (2006). "Bundling, unbundling, and pricing of multi-form products: The case of magazine content". In: *Journal of Interactive Marketing* 20.2, pp. 21–40.

Venkatesh, R. and Wagner Kamakura (2003). "Optimal Bundling and Pricing under a Monopoly: Contrasting Complements and Substitutes from Independently Valued Products". In: *Journal of Business* 76.2, pp. 211–232. URL: `http://econpapers.repec.org/RePEc:ucp:jnlbus:v:76:y:2003:i:2:p:211-232`.

Venkatesh, R. and Vijay Mahajan (2009). "The Design and Pricing of Bundles: a Review of Normative Guidelines and Practical Approaches". In: *Handbook of Pricing Research in Marketing*. Ed. by Vithala Rao. Northampton, MA: Edward Elgar Publishing, Inc.1.

Wallenstein, Andrew (2013). "Time to Free HBO Go From Its TV Shackles". In: *Variety* March 26. URL: `http://variety.com/2013/digital/news/time-to-free-hbo-go-from-its-tv-shackles-1200329375/`.